Vol. 05, No. 02, pp. 49 –54 (2024)
ISSN: 2709-6904

JoCEF

JOURNAL OF CIVIL ENGINEERING FRONTIERS

www.jocivilef.org

Deriving Ritz formulation for the Static Flexural Solutions of Sinusoidal Shear Deformable Beams

Benjamin Okwudili Mama[1], Onyedikachi Aloysius Oguaghamba[1], Charles Chinwuba Ike[2,*]

[1]*Department of Civil Engineering, University of Nigeria, Nsukka, Enugu State, Nigeria, (benjamin.mama@unn.edu.ng, aloysius.oguaghamba@unn.edu)*

[2]*Department of Civil Engineering, Enugu State University of Science and Technology, Agbani, Enugu State, Nigeria, charles.ike@esut.edu.ng*

Corresponding email: charles.ike@esut.edu.ng

Abstract

This paper presents the Ritz variational method for the static bending analysis of sinusoidal shear deformable beams. The theory accounts for transverse shear deformation and satisfies transverse shear stress-free conditions at the top and bottom surfaces of the beam. The total potential energy functional for the thick beam bending problem is formulated and minimized using a Ritz procedure. The problem considered simply supported boundary conditions and two cases of loading – uniformly distributed load over the span and point load at the center. The function is a function of two unknown displacement functions constructed in terms of unknown generalized displacement parameters and shape functions that satisfy the boundary conditions. Ritz minimization of the functional is used to find the generalized displacement parameters and then the displacements $w(x)$ and The displacements and stresses are found for the loading distributions considered. It is found that the results obtained agree remarkably well with the exact results obtained using the theory of elasticity. The differences between the present results and the exact solutions are less than 0.3% for maximum transverse displacement for both cases of loading considered. For uniform load over the beam, the result for l/t = 10 is 0.112% greater than the exact solution. For the point load at the center, the result for l/t = 10 is 4.468% greater than the exact solution. The increased difference in the point load case is due to the singular nature of the point load problem.

Keywords: *Trigonometric shear deformation beam theory, Ritz variational method, total potential energy functional, stress field, displacement field, thick beam*

Received: June 03, 2024 / Accepted: September 10, 2024 / Online: September 18, 2024

I. INTRODUCTION

The Euler-Bernoulli beam flexure theory was formulated using the hypothesis of orthogonality of the plane cross-section of the beam to the neutral axis before and after bending deformation [1 – 4]. The implication is that warping of the cross-section and transverse shear deformation are disregarded in the formulation, limiting the scope of use to thin beams where transverse shear strains make insignificant contribution to the flexural behaviour [5 – 7]. Beam theories have been derived to account for transverse shear deformation effects by Timoshenko [8], Levinson [9], Ghugal and Shimpi [10], Shimpi et al [11], Heyliger and Reddy [12] and others. Ghugal [13] has presented closed form solutions for the flexural problem of thick beams using the two dimensional theory of elasticity. Timoshenko and Goodier [14] derived the exact solutions to the bending problems of thick beams using the mathematical theory of elasticity. Other contributors to thick beam theory are: Ghugal and Sharma [15], Ghugal and Dahake [16], Sayyad [17, 18], Reddy [19], Ambartsumyam [20], Kruszewski [21] and Akavci [22].

Ike [23] used the Fourier series method (FSM) to develop accurate solutions for the displacements and stresses in hyperbolic shear deformable thick beams (HSDTB) subjected to distributed transverse loads. The HSDTB equations formulated in the study satisfied the transverse shear stress-free boundary conditions at the top and bottom surfaces of the

doi:10.38094/jocef50284

beam, and thus did not require shear correction factors. The results for displacements and stresses obtained in the work were accurate and compared well with previous solutions in the literature. In another work, Ike [24] developed an analytical solution to the buckling of thick beams based on a cubic polynomial shear deformation beam theory (CPSDBT). The CPSDBT satisfied the transverse shear stress-free boundary conditions, and thus did not need shear correction factors. The work gave buckling load solutions that were accurate and in agreement with previous solutions in the literature. The work did not however consider bending analysis of thick beam problems.

Mama et al [25], in a similar study, used Ritz variational method (RVM) for the bending analysis of thick beams formulated using trigonometric shear deformation beam bending theory (TSDBBT). Their formulation satisfied the transverse shear stress-free boundary conditions at the top and bottom surfaces. Their RVM results for stresses and displacements agreed with previous studies.

This work presents the variational formulation of the thick beam bending problem that accounts for transverse shear deformation effects using the sinusoidal shear deformation theory. Thereafter, it presents the Ritz direct method for solving the resulting formulation for the case of simple supports at the ends and for the two cases of uniformly distributed loading and point load applied at the midspan of the beam.

II. Formulation Of The Ritz Variational Functional For The Trigonometric Shear Deformable Beam

The paper considers homogeneous, elastic, isotropic beams subjected to transverse loads as shown in Figure 1.

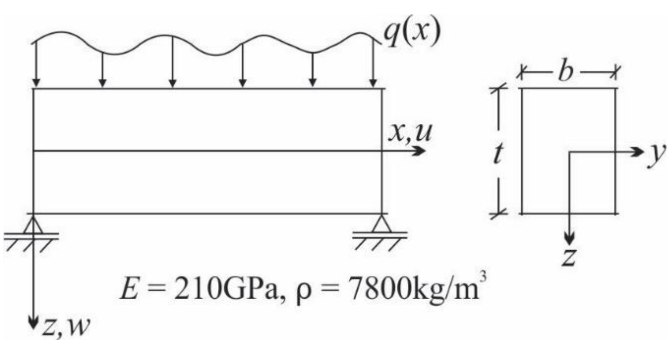

Fig. 1. Deep rectangular beam subjected to arbitrary loading q(x)

The axial displacement $u(x, z)$ and transverse displacement $w(x, z)$ of trigonometric shear deformable beams are:

$$u(x,z) = u_b(x,z) + u_s(x,z) = -z\frac{dw}{dx} + \left(\frac{t}{\pi}\sin\frac{\pi z}{t}\right)\theta(x) \quad (1)$$

where $\theta(x)$ is the warping function, u_b is the bending component and u_s is the shear component of $u(x, z)$.

$$w(x,z) = w(x) \quad (2)$$

The normal strain ε_{xx} and shear strains γ_{xz} are:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{d^2w}{dx^2} + \left(\frac{t}{\pi}\sin\frac{\pi z}{t}\right)\frac{d\theta(x)}{dx} \quad (3)$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \theta(x)\cos\frac{\pi z}{t} \quad (4)$$

The normal and transverse stresses are found from the stress-strain relations as:

$$\sigma_{xx} = E\varepsilon_{xx} = -Ez\frac{d^2w}{dx^2} + \frac{Et}{\pi}\sin\frac{\pi z}{t}\frac{d\theta(x)}{dx} \quad (5)$$

$$\tau_{xz} = G\gamma_{xz} = G\theta(x)\cos\frac{\pi z}{t} \quad (6)$$

where σ_{xx} is the normal stress, τ_{xz} is the transverse stress.

A. Total Potential Energy Functional Π

The total potential energy functional for the thick beam flexure problem is found as:

$$\Pi = \frac{1}{2}\int_{-b/2}^{b/2}\int_0^l\int_{-t/2}^{t/2}(\sigma_{xx}\varepsilon_{xx} + \tau_{xz}\gamma_{xz})dydxdz - \int_0^l q(x)w(x)dx \quad (7)$$

$$\Pi = \frac{b}{2}\int_0^l\int_{-t/2}^{t/2}(E\varepsilon_{xx}^2 + G\gamma_{xz}^2)dxdz - \int_0^l q(x)w(x)dx \quad (8)$$

Hence,

$$\Pi = \frac{b}{2}\int_0^l\int_{-t/2}^{t/2}\left\{E\left(-z\frac{d^2w}{dx^2} + \frac{t}{\pi}\sin\frac{\pi z}{t}\frac{d\theta(x)}{dx}\right)^2 + G\left(\theta(x)\cos\frac{\pi z}{t}\right)^2\right\}dxdz - \int_0^l q(x)w(x)dx \quad (9)$$

Further simplification gives:

$$\Pi = \frac{1}{2}\int_0^l\left\{F_1\left(\frac{d^2w}{dx^2}\right)^2 + F_2\left(\frac{d\theta}{dx}\right)^2 + F_3\frac{d^2w}{dx^2}\frac{d\theta}{dx} + F_4(\theta(x))^2 - 2q(x)w(x)\right\}dx \quad (10)$$

wherein

$$F_1 = \int_{-t/2}^{t/2} bEz^2dz = EI \quad (11)$$

$$F_2 = \int_{-t/2}^{t/2} b\frac{Et^2}{\pi^2}\sin^2\frac{\pi z}{t}dz = \frac{Ebt^3}{2\pi^2} = \frac{6EI}{\pi^2} \quad (12)$$

$$F_3 = -2\int_{-t/2}^{t/2} bE\frac{zt}{\pi}\sin\frac{\pi z}{t}dz = -\frac{4Ebt^3}{\pi^3} = -\frac{48EI}{\pi^3} \quad (13)$$

$$F_4 = \int_{-t/2}^{t/2} Gb\cos^2\frac{\pi z}{t}dz = \frac{GA}{2} \quad (14)$$

Hence

$$\Pi = \frac{1}{2}\int_0^l\left[EI\left(\frac{d^2w}{dx^2}\right)^2 + \frac{6EI}{\pi^2}\left(\frac{d\theta}{dx}\right)^2 - \frac{48EI}{\pi^3}\left(\frac{d^2w}{dx^2}\frac{d\theta}{dx}\right)\right.$$

$$+\frac{GA}{2}(\theta(x))^2 - 2q(x)w(x)\Bigg]dx$$
$$= \Pi\left(x, w(x), \theta(x), w''(x), \theta'(x)\right) \quad (15)$$

III. METHODOLOGY

The paper considers thick beams with simply supported ends as shown in Figure 1. The boundary conditions are:

$$w(0) = w(l) = 0$$
$$w''(0) = w''(l) = 0 \quad (16)$$
$$\theta'(0) = \theta'(l) = 0$$

Hence $w(x)$ and $\theta(x)$ that satisfy the boundary conditions can be expressed using the infinite series:

$$w(x) = \sum_{i=1}^{\infty} w_i \sin\frac{i\pi x}{l} \quad (17)$$

$$\theta(x) = \sum_{i=1}^{\infty} \theta_i \cos\frac{i\pi x}{l} \quad (18)$$

wherein w_i and θ_i are generalized displacement parameters for $w(x)$ and $\theta(x)$ respectively.

The static transverse load is similarly expressed in infinite Fourier sine series form as:

$$q(x) = \sum_{i=1}^{\infty} q_1 \sin\frac{i\pi x}{l} \quad (19)$$

where

$$q_i = \frac{2}{l}\int_0^l q(x)\sin\frac{i\pi x}{l}dx \quad (20)$$

q_i is the Fourier series coefficient of $q(x)$

The functional to be minimized becomes expressed in terms of w_i and θ_i as:

$$\Pi = \frac{1}{2}\int_0^l\Bigg[EI\left(\sum_{i=1}^{\infty}\left(-\frac{i\pi}{l}\right)^2 w_i \sin\frac{i\pi x}{l}\right)^2 +$$
$$\frac{6EI}{\pi^2}\left(\sum_{i=1}^{\infty}\left(-\frac{i\pi}{l}\right)\theta_i \sin\frac{i\pi x}{l}\right)^2 -$$
$$\frac{48EI}{\pi^3}\left(\sum_{i=1}^{\infty}\left(-\frac{i\pi}{l}\right)^2 w_i \sin\frac{i\pi x}{l}\right)\left(\sum_{i=1}^{\infty}\left(\frac{i\pi}{l}\right)\theta_i \sin\frac{i\pi x}{l}\right)+$$
$$\frac{GA}{2}\left(\sum_{i=1}^{\infty}\theta_i \cos\frac{i\pi x}{l}\right)^2 -$$
$$2\left(\sum_{i=1}^{\infty}q_i \sin\frac{i\pi x}{l}\right)\left(\sum_{i=1}^{\infty}w_i \sin\frac{i\pi x}{l}\right)\Bigg]dx \quad (21)$$

For equilibrium of the thick beam bending problem, Π is minimized with respect to the unknown generalized displacements w_i and θ_i. Thus, for minimization of Π, we require;

$$\frac{\partial\Pi}{\partial w_i} = 0 \quad (22)$$

$$\frac{\partial\Pi}{\partial\theta_i} = 0 \quad (23)$$

These conditions yield after simplification:

$$\begin{pmatrix} EI\left(\frac{i\pi}{l}\right)^4 & -\frac{24}{\pi^3}EI\left(\frac{i\pi}{l}\right)^3 \\ -\frac{24}{\pi^3}EI\left(\frac{i\pi}{l}\right)^3 & \left(\frac{6EI}{\pi^2}\left(\frac{i\pi}{l}\right)^2 + \frac{GA}{2}\right) \end{pmatrix}\begin{pmatrix} w_i \\ \theta_i \end{pmatrix} = \begin{pmatrix} q_i \\ 0 \end{pmatrix} \quad (24)$$

By Cramer's rule

$$w_i = \frac{\Delta_{11}}{\Delta_{00}} \quad (25)$$

$$\theta_i = \frac{\Delta_{22}}{\Delta_{00}} \quad (26)$$

$$\Delta_{00} = \begin{vmatrix} EI\left(\frac{i\pi}{l}\right)^4 & -\frac{24}{\pi^3}EI\left(\frac{i\pi}{l}\right)^3 \\ -\frac{24}{\pi^3}EI\left(\frac{i\pi}{l}\right)^3 & \left(\frac{6EI}{\pi^2}\left(\frac{i\pi}{l}\right)^2 + \frac{GA}{2}\right) \end{vmatrix} \quad (27)$$

$$\Delta_{11} = \begin{vmatrix} q_i & -\frac{24}{\pi^3}EI\left(\frac{i\pi}{l}\right)^3 \\ 0 & \left(\frac{6EI}{\pi^2}\left(\frac{i\pi}{l}\right)^2 + \frac{GA}{2}\right) \end{vmatrix} \quad (28)$$

$$\Delta_{22} = \begin{vmatrix} EI\left(\frac{i\pi}{l}\right)^4 & q_i \\ -\frac{24}{\pi^3}EI\left(\frac{i\pi}{l}\right)^3 & 0 \end{vmatrix} \quad (29)$$

Hence,

$$w_i = \frac{q_i\left(\frac{6EI}{\pi^2}\left(\frac{i\pi}{l}\right)^2 + \frac{GA}{2}\right)}{EI\left(\frac{i\pi}{l}\right)^4\left(\frac{6EI}{\pi^2}\left(\frac{i\pi}{l}\right)^2 + \frac{GA}{2}\right) - \left(\frac{24EI}{\pi^3}\left(\frac{i\pi}{l}\right)^3\right)^2} \quad (30)$$

$$\theta_i = \frac{q_i\left(\frac{24EI}{\pi^3}\left(\frac{i\pi}{l}\right)^3\right)}{EI\left(\frac{i\pi}{l}\right)^4\left(\frac{6EI}{\pi^2}\left(\frac{i\pi}{l}\right)^2 + \frac{GA}{2}\right) - \left(\frac{24EI}{\pi^3}\left(\frac{i\pi}{l}\right)^3\right)^2} \quad (31)$$

A. *Axial Displacement u(x, z)*

Axial displacement is found as:

$$u(x,z) = \sum_{i=1}^{\infty}\left(-z\left(\frac{i\pi}{l}\right)w_i + \left(\frac{t}{\pi}\sin\frac{\pi z}{t}\right)\theta_i\right)\cos\frac{i\pi x}{l} \quad (32)$$

B. *Transverse Displacement w(x)*

The transverse displacement $w(x)$ is found at $x = l/2$ as

$$w(x = l/2, z) = \sum_{i=1}^{\infty} w_i \sin\frac{i\pi}{2} \quad (33)$$

Axial bending stress σ_{xx} is formed as:

$$\sigma_{xx} = E\sum_{i=1}^{\infty}\left(z\left(\frac{i\pi}{l}\right)^2 w_i - \left(\frac{t}{\pi}\sin\frac{\pi z}{t}\right)\frac{i\pi}{l}\theta_i\right)\sin\frac{i\pi x}{l} \quad (34)$$

Hence,

$$\sigma_{xx}\left(x = l/2, z = \pm t/2\right) = E\sum_{i=1}^{\infty}\left(\pm\frac{t}{2}\left(\frac{i\pi}{l}\right)^2 w_i - \frac{t}{\pi}\sin\left(\frac{\pm\pi}{2}\right)\frac{i\pi}{l}\theta_i\right)\sin\frac{\pi}{2} \quad (35)$$

The transverse shear stress is:

$$\tau_{xz} = G\sum_{i=1}^{\infty}\left(\cos\frac{\pi z}{t}\right)\theta_i \cos\frac{i\pi x}{l} \quad (36)$$

$$\tau_{xz}(x = 0, z = 0) = \sum_{i=1}^{\infty} G\theta_i \quad (37)$$

IV. Results

The results are presented for the thick beam bending problem with material parameters E = 210GPa, $\rho = 7800\text{kg/m}^3$ for simply supported ends and for two cases of loading – namely uniformly distributed load over the entire span and point load at the center.

For uniformly distributed load of intensity q_0 over the entire beam span,

$$q_i = \frac{4q_0}{i\pi} \qquad i = 1, 3, 5, 7, \ldots$$
$$q_i = 0 \qquad i = 2, 4, 6, 8, \ldots \quad (38)$$

For point load P_0 at $x = \varepsilon$,

$$q_i = \frac{2P}{l}\sin\frac{i\pi\varepsilon}{l} \quad (39)$$

where ε is the distance of the point load from the origin.

For point load at the center, $\varepsilon = l/2$ and

$$q_i = \frac{2P}{l}\sin\frac{i\pi}{2} \quad (40)$$

The results are presented using non-dimensional representations for displacements and stresses defined as follows:

$$w = \bar{w}\frac{q_0 l^4}{10Ebh^2}$$
$$\sigma_{xx} = \bar{\sigma}_{xx}\frac{q_0}{b} \quad (41)$$
$$\tau_{xz} = \bar{\tau}_{xz}\frac{q_0}{b}$$

$\bar{w}$, $\bar{\sigma}_{xx}$, $\bar{\tau}_{xz}$ are dimensionless displacement, normal and shear stresses respectively.

A measure of the deviation of the results from the two-dimensional (2D) theory of elasticity solution is obtained as follows:

$$\%\text{Difference} = \left\{\frac{\text{results} - \text{Exact result (2D elasticity result)}}{\text{Exact result (2D elasticity result)}}\right\}\times 100\%$$

(42)

The results obtained for the cases of uniformly distributed load and point load applied at the center are shown in Tables 1, 2, 3, 4 which also presents previous results from other studies.

V. Discussion

This paper has presented the Ritz variational method for the flexural analysis of thick simply supported beams described using the trigonometric shear deformation theory. The transverse loads considered are uniformly distributed loads over the entire beam span and point load acting at the center of the span. The functional for the thick beam bending problem was formulated as a function of two unknown functions and derivatives of the two unknown functions – Equation (15). The unknown functions to be found are considered in the form of single infinite series given by Equations (17) and (18) in terms of unknown generalized displacement parameters. The displacement functions are constructed to apriori satisfy all the conditions at the simply supported ends. The unknown functions are found using the condition for the extremum of the Ritz functional – Equations (22) and (23).

Table 1 shows that for uniformly distributed load the present results for $w(x = l/2, z = 0)$ when l/t = 10 is 0.187% different from the exact theory of elasticity solution which illustrates that the present result is more accurate than previous results by Reddy (0.25% difference), Timoshenko (0.25% difference), and Euler-Bernoulli (–2.190% difference). Similarly, Table 2 shows that for uniformly distributed loads, present results for $\sigma_{xx}(x = l/2, z = \pm t/2)$ for l/t = 10 is just 0.112% different from the Timoshenko and Goodier [14] exact solution obtained using the theory of elasticity. The result is more accurate than Timoshenko and Euler-Bernoulli results.

Table 3 which presents the results for thick isotropic beam subjected to point load at the center of the span shows that present result for $w(x = l/2, z = 0)$ for l/t = 10 is 0.288% different from the Timoshenko and Goodier solutions [14]. The table further shows the present method gives more accurate results then the Reddy, Timoshenko and Euler-Bernoulli results.

Table 4 presents the results for thick isotropic beam subjected to point load at the midspan. The present results for $\sigma_{xx}(x = l/2, z = 0)$ for l/t = 10 is 4.468% different from the exact theory of elasticity solution obtained by Timoshenko and Goodier [14].

VI. Conclusion

1. The axial displacement $u(x, z)$ is obtained as a convergent cosine series with infinite terms.
2. The transverse displacement $w(x)$ is maximum at $x = l/2$, $z = 0$ and is expressed as a convergent sine series with infinite terms.
3. The axial bending stress σ_{xx} is obtained as a convergent infinite sine series.
4. Maximum value of σ_{xx} is obtained at $x = l/2$, $z = \pm t/2$.
5. The transverse shear τ_{xz} is obtained as an infinite cosine series. Maximum transverse shear stress τ_{xz} occurs at $x = 0$, $z = 0$ and is given in series form.
6. The trigonometric shear deformation theory is variationally consistent, and transverse shear stress free conditions are satisfied at the beam surfaces, removing the need for transverse shear stress modification factors.

TABLE I. COMPARISON OF SOLUTIONS FOR AXIAL AND TRANSVERSE DISPLACEMENT FOR THICK ISOTROPIC BEAMS SUBJECTED TO UNIFORMLY DISTRIBUTED LOAD FOR DIFFERENT ASPECT RATIOS

l/t	Theory	$\bar{u}(x = l, z = \pm t/2)$	% Difference	$\bar{w}(x = l/2, z = 0)$	% Difference
2	Present	2.259	2.682	2.529	3.098
	Reddy [9]	2.245	2.045	2.532	3.221
	Timoshenko [8]	2.000	–9.091	2.538	3.465
	Euler-Bernoulli	2.000	–9.091	1.563	–36.282
	Ambartsumyan [20]	–	–	2.357	–3.913
	Kruszewski [21]	–	–	2.215	2.527
	Akavci [22]	–	–	2.523	2.853
	Timoshenko and Goodier [14]	2.200	0	2.453	0
4	Present	16.535	4.652	1.805	1.120
	Reddy [9]	16.504	4.456	1.806	1.176
	Timoshenko [8]	16.000	1.265	1.806	1.176
	Euler-Bernoulli	16.000	1.265	1.563	–12.437
	Ambartsumyan [20]	–	–	1.762	–1.288
	Kruszewski [21]	–	–	1.805	1.120
	Akavci [22]	–	–	1.804	1.064
	Timoshenko and Goodier [14]	15.800	0	1.785	0
10	Present	251.35	0.745	1.601	0.187
	Reddy [9]	251.27	0.709	1.602	0.25
	Timoshenko [8]	250.00	0.200	1.602	0.25
	Euler-Bernoulli	250.00	0.200	1.563	–2.190
	Ambartsumyan [20]	–	–	1.595	–0.187
	Kruszewski [21]	–	–	1.602	0.25
	Akavci [22]	–	–	1.601	0.187
	Timoshenko and Goodier [14]	249.50	0	1.598	0

TABLE II. COMPARISON OF SOLUTIONS FOR AXIAL BENDING STRESS σ_{xx} AT $(x = l/2, z = \pm t/2)$ AND TRANSVERSE SHEAR STRESS τ_{xz} $(x = 0, z = 0)$ FOR THICK ISOTROPIC BEAM SUBJECTED TO UNIFORMLY DISTRIBUTED LOAD FOR VARIOUS ASPECT RATIOS

l/t	Theory	$\bar{\sigma}_{xx}$	% Diff	τ_{xz}^{CR}	% Diff	τ_{xz}^{EE}	% Diff
2	Present	3.278	2.438	1.451	–3.267	1.250	–16.667
	Reddy [19]	3.261	1.960	1.415	–5.667	1.262	–15.867
	Timoshenko [8]	3.000	–6.25	0.984	–34.40	1.477	–1.533
	Ambartsumyan [20]	3.210	0.312	1.156	–22.93	–	–
	Kruszewski [21]	3.261	1.906	1.333	–11.13	–	–
	Akavci [22]	3.253	1.656	1.397	–6.866	–	–
	Euler-Bernoulli	3.000	–6.25	–	–	1.477	–1.533
	Timoshenko and Goodier [14]	3.200	0	1.500	0	1.500	0
4	Present	12.280	0.656	2.993	–0.233	2.783	–7.233
	Reddy [19]	12.263	0.516	2.908	–3.067	2.795	–6.833
	Timoshenko [8]	12.000	–1.693	1.969	–34.367	2.953	–1.567
	Ambartsumyan [20]	12.212	0.098	2.389	–20.36	–	–
	Kruszewski [21]	12.262	0.508	2.836	–5.466	–	–
	Akavci [22]	12.254	0.442	2.882	–3.933	–	–
	Euler-Bernoulli	12.00	–1.693	–	–	2.953	–1.567
	Timoshenko and Goodier [14]	12.20	0	3.000	0	3.00	0
10	Present	75.284	0.112	7.591	1.213	7.295	–2.733
	Reddy [19]	75.268	0.090	7.361	–1.853	7.304	–2.61
	Timoshenko [8]	75.00	–0.264	4.922	–34.373	7.383	–1.56
	Ambartsumyan [20]	75.216	0.021	6.066	–19.12	–	–
	Kruszewski [21]	75.266	0.088	7.328	–2.293	–	–
	Akavci [22]	75.259	0.078	7.312	–2.506	–	–
	Euler-Bernoulli	75.00	–0.264	–	–	7.383	–1.56
	Timoshenko and Goodier [14]	75.20	0	7.500	0	7.500	0

TABLE III. COMPARISON OF RESULTS FOR AXIAL DISPLACEMENT $\bar{u}$ AT $(x = l, z = \pm t/2)$, TRANSVERSE DISPLACEMENT $\bar{w}$ AT $(x = l/2, z = 0)$ FOR THICK ISOTROPIC BEAMS SUBJECTED TO POINT LOAD AT THE CENTER FOR VARIOUS ASPECT RATIOS

l/t	Theory/Reference	$\bar{u}$	% Difference	$\bar{w}$	% Difference
2	Present method	3.2776	–	4.3257	7.63
	Reddy [19]	3.2611	–	4.3399	7.899
	Timoshenko [8]	3.0001	–	4.4198	9.978
	Euler-Bernoulli	3.0001	–	2.5000	–37.792
	Timoshenko and Goodier [14]	–	–	4.0188	0
4	Present method	24.5591	–	2.9706	1.9948
	Reddy [19]	24.5263	–	2.9726	2.0635
	Timoshenko [8]	24.0007	–	2.9799	2.3142
	Euler-Bernoulli	24.0007	–	2.5000	–14.1631
	Timoshenko and Goodier [14]	–	–	2.9125	0
10	Present method	376.4214	–	2.5764	0.288
	Reddy [19]	376.3385	–	2.5765	0.292
	Timoshenko [8]	375.0122	–	2.5768	0.304
	Euler-Bernoulli	375.0109	–	2.5000	–2.686
	Timoshenko and Goodier [14]	–	–	2.5690	0

TABLE IV. COMPARISON OF RESULTS FOR AXIAL BENDING STRESS $\bar{\sigma}_{xx}$ AT $(x = l/2, z = \pm t/2)$, TRANSVERSE SHEAR STRESS $\bar{\tau}_{xz}^{CR}$ $(x = 0, z = 0)$, τ_{xz}^{EE} AT $(x = 0, z = 0)$ FOR VARIOUS ASPECT RATIOS FOR POINT LOAD AT THE CENTER OF SPAN

l/t	Theory/Reference	$\bar{\sigma}_{xx}$	% Diff	τ_{xz}^{CR}	% Diff	τ_{xz}^{EE}	% Diff
2	Present method	9.3101	67.907	1.5532	–	1.4347	–
	Reddy [19]	9.3469	68.571	1.5059	–	1.4290	–
	Timoshenko [8]	5.9065	6.523	1.0244	–	1.5367	–
	Euler-Bernoulli	5.9065	6.523	–	–	1.5367	–
	Timoshenko and Goodier [14]	5.5448	0	–	–	–	–
4	Present method	28.7619	12.838	3.1253	4.177	2.9283	–
	Reddy [19]	28.6790	12.513	3.0319	1.063	2.9284	–
	Timoshenko [8]	23.6261	–7.311	2.0489	–31.703	3.0733	–
	Euler-Bernoulli	23.6261	–7.311	–	–	3.0733	–
	Timoshenko and Goodier [14]	25.4896	0	3.0000	0	–	–
10	Present method	154.3242	4.468	7.8912	5.216	7.5636	–
	Reddy [19]	154.0091	4.255	7.6519	2.025	7.5733	–
	Timoshenko [8]	147.6634	–0.041	5.1223	–31.702	7.6834	–
	Euler-Bernoulli	147.6630	–0.0412	–	–	7.6834	–
	Timoshenko and Goodier [14]	147.7239	0	7.500	0	–	–

REFERENCES

[1] C.C. Ike and E.U. Ikwueze, "Ritz method for the analysis of statically indeterminate Euler-Bernoulli beams," Saudi Journal of Engineering and Technology, Vol. 3, Issue 3, pp. 133 – 140, 2018.

[2] C.C. Ike and E.U. Ikwueze, "Fifth degree Hermittian polynomial shape functions for the finite element analysis of clamped simply supported Euler-Bernoulli beam," American Journal of Engineering Research, Vol. 7, Issue 4, pp. 97 – 105, 2018.

[3] C.C. Ike, "Fourier sine transform method for the free vibration of Euler-Bernoulli beam resting on Winkler foundation," International Journal of Darshan Institute on Engineering Research and Emerging Technologies (IJDI-ERET), Vol. 7, No. 1, pp. 1 – 6, 2018.

[4] C.C. Ike, "Point collocation method for the analysis of Euler-Bernoulli beam on Winkler foundation," International Journal of Darshan Institute on Engineering Research and Emerging Technologies (IJDI-ERET), Vol. 7, No. 2, pp 1 – 7, 2018.

[5] C.C. Ike, "Timoshenko beam theory for the flexural analysis of moderately thick beams – variational formulation and closed form solutions," Tecnica Italiana – Italian Journal of Engineering Science, Vol. 63, No. 1, pp 34 – 45, 2019.

[6] H.N. Onah, C.U. Nwoji, M.E. Onyia, B.O. Mama, and C.C. Ike, "Exact solutions for the elastic buckling problem of moderately thick beams," Revue des Composites et des Materiaux Avances, Vol. 30, No. 2, pp 83 – 93, 2020.

[7] C.C. Ike, C.U. Nwoji, B.O. Mama, H.N. Onah, and M.E. Onyia, "Laplace transform method for the elastic buckling analysis of moderately thick beams," International Journal of Engineering Research and Technology, Vol. 12, No. 10, pp 1626 – 1638, 2019.

[8] S.P. Timoshenko, "On the correction for shear of the differential equation for transverse vibration of prismatic bars," Philosophical Magazine, Vol. 41 No. 6, pp. 742 – 746, 1921.

[9] M. Levinson, "A new rectangular beam theory," Journal of Sound and Vibration, Vol. 74 No. 1, pp. 81 – 87, 1981.

[10] Y.M. Ghugal and R.P. Shimpi, "A review of refined shear deformation theories for isotropic and anisotropic laminated beams," Journal of Reinforced Plastics and Composites, Vol. 20 No. 3, pp. 255 – 272, 2001.

[11] R.P. Shimpi, P.J. Guruprasad, and K.S. Pakhare, "Simple two variable refined theory for shear deformable isotropic rectangular beams," Journal of Applied and Computational Mechanics, Vol. 6 No. 3, pp. 394 – 415, 2020.

[12] P.R. Heyliger and J.N. Reddy, "A higher order beam finite element for bending and vibration problems," Journal of Sound and Vibration, Vol. 126 No. 2, pp. 309 – 326, 1988.

[13] Y. Ghugal, A two-dimensional exact elasticity solution of thick beams. Departmental Report – 1. Department of Applied Mechanics, Government Engineering College, Aurangabad, India, pp 1 – 96, 2006.

[14] S.P. Timoshenko and J.N. Goodier, Theory of Elasticity, 3rd International Edition. Singapore: McGraw Hill, 1970.

[15] Y.M. Ghugal and R. Sharma, "A refined shear deformation theory for flexure of thick beams," Latin American Journal of Solids and Structures, Vol. 8, pp. 183 – 195, 2011.

[16] Y.M. Ghugal and A.G. Dahake, "Flexural analysis of deep beam subjected to parabolic loads using refined shear deformation theory," Applied Computational Mechanics, Vol 6 No. 2, pp 163 – 172, 2012.

[17] A.S. Sayyad, "Comparison of various shear deformation theories for the free vibration of thick isotropic beams," Latin American Journal of Solids and Structures, Vol. 2 No 1, pp. 85 – 97, 2011.

[18] A.S. Sayyad, "Comparison of various refined beam theories for the bending and free vibration analysis of thick beams," Applied and Computational Mechanics, Vol. 5, pp. 217 – 230, 2011.

[19] J.N. Reddy, "A general non-linear third order theory of plates with moderate thickness," International Journal of Non-linear Mechanics, Vol. 25 No. 6, pp. 677 – 686, 1990.

[20] S.A. Ambartsumyan, "On the theory of bending of plates," Izv Otd Tech Nauk ANSSSR, Vol. 5, pp. 67 – 77, 1958.

[21] E.T. Kruszewski, Effect of transverse shear and rotary inertia on the natural frequency of a uniform beam. National Advisory Committee for Aeronautics Technical Note 1909 (NACA-TN-1909), pp. 1 – 16, 1949.

[22] S.S. Akavci, "Buckling and free vibration analysis of symmetric and antisymmetric laminated composite plates on an elastic foundation," Journal of Reinforced Plastics and Composites, Vol. 26 No 18, pp. 1907 – 1919, 2007.

[23] C. C. Ike "An analytical solution to buckling of thick beams based on a cubic polynomial shear deformation beam theory" Engineering and Technology Journal Vol 42, No 1, pp 90-103, 2024.

[24] C. C. Ike "Fourier series method for finding displacements and stress fields in hyperbolic shear deformable thick beams subjected to distributed transverse loads" Journal of Computational Applied Mechanics, Vol 53, No 1, pp 126-141, 2022.

[25] B. O. Mama, O. A. Oguaghamba, C. C. Ike "Ritz variational method for the bending analysis of trigonometric shear deformable beams" Proceedings Engineering and Sustainable Development Conference University of Nigeria Nsukka 2021 (ESDCUNN 2021) pp 47, 3rd-5th February 2021.